

Renormalization of noncommutative $U(N)$ gauge theories

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Abstract

We give an explicit proof that noncommutative $U(N)$ gauge theories are one-loop renormalizable.

1 Introduction

Since [1] it has been clear that the concept of space in the presence of nearby D-branes is radically modified. One way to express it is to say that the positions of the branes are replaced by suitable matrices, whose entries actually represent open strings stretched among them. This entails in particular that at short brane distances space becomes noncommutative. This picture is indeed very suggestive, but not very effective in representing space non-commutativity. It has been only recently, [2, 3, 4, 5, 6], that noncommutativity has surfaced in a very effective and manageable way. This happens precisely when D-branes are in presence of a constant NSNS B-field. In this case, the low energy effective action of the open strings attached to the branes can be represented by a Euclidean field theory defined on a noncommutative spacetime. All this clearly holds at a semiclassical level (i.e. tree amplitudes computed in the string and the field theory setting compare well). However one can try to compare loop amplitudes calculated both in string theory and in the corresponding noncommutative field theory, in order to see how effective the noncommutative effective field theory is. Several calculations of this type have been carried out, see [7, 8, 9, 10, 12, 13, 14, 15, 16, 17, 18, 21, 19, 20, 22] and in particular [11, 13, 26], and some discrepancies have recently surfaced, [27, 28]. The latter papers seem to imply that more general \star products are necessary, in order for the effective field theory to faithfully represent string loop contributions.

It seems to be important therefore to know exactly what are the properties of a noncommutative YM theory we can rely on. One of the basic properties is renormalizability. In this paper we consider a noncommutative YM theory with $U(N)$ gauge group in 4D without matter, and study its one-loop renormalizability properties. Since we take for granted that non-planar singularities are damped by the noncommutative parameter θ [7, 23, 24, 25], we only consider the planar one-loop contributions. This has already been partially done as far as the $U(1)$ gauge theory is concerned [11, 13], and for two- and three-point functions for $U(N)$, [26]. We want to complete such calculations by computing the one-loop four-point function in a pure noncommutative $U(N)$ Yang-Mills theory in 4D, for which some doubts have been raised. It is obvious that if gauge (BRST) invariance is assumed, the calculation we do is superfluous. But in a noncommutative gauge theory some caution has to be used and BRST invariance has to be proved rather than assumed. We have in fact a simple counterexample: NCSO(2) gauge theory, which was introduced in [32]. If we define the Feynman rules in the most obvious way, it is immediate to see that for 2- and 3-gluon vertices the one-loop corrections identically vanish, while the 4-gluon has a divergent part at one-loop. Therefore this theory is not one-loop renormalizable (at least in the ordinary sense).

Our final result is however reassuring. *Noncommutative $U(N)$ gauge theories are one-loop renormalizable.*

The paper is organized as follows. In the next section we set notations and conventions. In section 3 we compute the relevant one-loop amplitudes for a noncommutative $U(N)$ theory. We also make a comment on the possibility of defining consistent noncommutative $SU(N)$ Feynman rules.

2 Notations, conventions and $u(N)$ tensors

Our noncommutative theory is specified by the action

$$S = \int d^4x \operatorname{Tr} \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\alpha} (\partial_\mu A^\mu)^2 + \frac{1}{2} (i\bar{c} * \partial_\mu D^\mu c - i\partial_\mu D^\mu c * \bar{c}) \right) \quad (2.1)$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - ig(A_\mu * A_\nu - A_\nu * A_\mu) \quad (2.2)$$

and the Moyal product is defined with respect to the parameter $\theta^{\mu\nu}$. The potential A_μ is valued in the Lie algebra $u(N)$, i.e. is an hermitian matrix, and we will choose the Feynman gauge $\alpha = 1$. As is customary in dealing with 4D field theories, throughout the paper we use a Minkowski formulation of the theory, although its brane origin is Euclidean.

Since the properties of the Lie algebra $u(N)$ tensors are crucial in our calculation, we devote the rest of this section to deriving them.

We use a basis t^a , $a = 1, \dots, N^2 - 1$ of traceless hermitean matrices for the Lie algebra $su(N)$, with normalization

$$\operatorname{tr}(t^a t^b) = \frac{1}{2} \delta^{ab} \quad (2.3)$$

and structure constants f_{abc} defined by

$$[t^a, t^b] = i f_{abc} t^c. \quad (2.4)$$

We define also the third order ad-invariant completely symmetric tensor d_{abc} by means of

$$\{t^a, t^b\} = \frac{1}{N} \delta_{ab} + d_{abc} t^c. \quad (2.5)$$

Next we pass to the Lie algebra $u(N)$ by introducing the additional generator $t^0 = \frac{1}{\sqrt{2N}} \mathbf{1}_N$. Corresponding to any index a for $su(N)$ we introduce the index $A = (0, a)$, so that A runs from 0 to $N^2 - 1$. We have

$$[t^A, t^B] = i f_{ABC} t^C, \quad \{t^A, t^B\} = d_{ABC} t^C \quad (2.6)$$

where f_{ABC} is completely antisymmetric, f_{abc} is the same as for $su(N)$ and $f_{0BC} = 0$, while d_{ABC} is completely symmetric; d_{abc} is the same as for $su(N)$, $d_{0BC} = \sqrt{\frac{2}{N}} \delta_{BC}$, $d_{00c} = 0$ and $d_{000} = \sqrt{\frac{2}{N}}$. We have also

$$\operatorname{Tr}(t^A t^B) = \frac{1}{2} \delta^{AB}. \quad (2.7)$$

The following identities hold and will be extensively used below

$$\begin{aligned} f_{ABX} f_{XCD} + f_{ACX} f_{XDB} + f_{ADX} f_{XBC} &= 0 \\ f_{ABX} d_{XCD} + f_{ACX} d_{XDB} + f_{ADX} d_{XBC} &= 0 \\ f_{ADX} f_{XBC} &= d_{ABX} d_{XCD} - d_{ACX} d_{XDB} \end{aligned} \quad (2.8)$$

Next we define the matrices F_A, D_A as follows

$$(F_A)_{BC} = f_{BAC}, \quad (D_A)_{BC} = d_{BAC} \quad (2.9)$$

In the evaluation of Feynman diagrams we need to know traces of two, three and four such matrices. We borrow from the literature, [29, 30, 31], the corresponding results for $su(N)$ and extend them to $u(N)$. Denoting by $\widehat{\text{Tr}}$ the traces over the relevant $N^2 \times N^2$ space, we obtain

$$\begin{aligned}\widehat{\text{Tr}}(F_A F_B) &= -N c_A \delta_{AB}, & c_A &= 1 - \delta_{A,0} \\ \widehat{\text{Tr}}(D_A D_B) &= N d_A \delta_{AB}, & d_A &= 2 - c_A \\ \widehat{\text{Tr}}(F_A D_B) &= 0\end{aligned}\tag{2.10}$$

$$\begin{aligned}\widehat{\text{Tr}}(F_A F_B F_C) &= -\frac{N}{2} f_{ABC} \\ \widehat{\text{Tr}}(F_A F_B D_C) &= -\frac{N}{2} d_{ABC} c_A c_B d_C \\ \widehat{\text{Tr}}(F_A D_B D_C) &= \frac{N}{2} f_{ABC} \\ \widehat{\text{Tr}}(D_A D_B D_C) &= \frac{N}{2} \eta_{ABC} d_{ABC}\end{aligned}\tag{2.11}$$

where $\eta_{ABC} = d_A d_B d_C - 4\delta_{A+B+C,0}$. Finally

$$\begin{aligned}\widehat{\text{Tr}}(F_A F_B F_C F_D) &= \left[\frac{1}{2} \delta_{(AB} \delta_{CD)} + \frac{N}{8} (d_{ABX} d_{CDX} + d_{ADX} d_{BCX}) \right. \\ &\quad \left. + \frac{N}{8} (f_{ADX} f_{BCX} - f_{ABX} f_{CDX}) \right] c_A c_B c_C c_D \\ \widehat{\text{Tr}}(F_A F_B F_C D_D) &= -\frac{N}{4} (d_{ABX} f_{CDX} + f_{ABX} d_{CDX}) c_A c_B c_C d_D \\ \widehat{\text{Tr}}(F_A F_B D_C D_D) &= c_A c_B \left[c_C c_D \frac{1}{2} (\delta_{AC} \delta_{BD} - \delta_{AB} \delta_{CD} + \delta_{AD} \delta_{BC}) \right. \\ &\quad \left. + \frac{N}{8} d_C d_D (f_{ABX} f_{CDX} - f_{ADX} f_{BCX} - d_{ABX} d_{CDX} - d_{ADX} d_{BCX}) \right] \\ \widehat{\text{Tr}}(F_A D_B F_C D_D) &= \frac{1}{2} (\delta_{AB} \delta_{CD} + \delta_{AD} \delta_{BC} - \delta_{AC} \delta_{BD}) c_A c_B c_C c_D \\ &\quad + \frac{N}{8} (f_{ABX} f_{CDX} - f_{ADX} f_{BCX} - d_{ABX} d_{CDX} - d_{ADX} d_{BCX}) c_A d_B c_C d_D \\ \widehat{\text{Tr}}(F_A D_B D_C D_D) &= \frac{N}{4} (f_{ABX} d_{CDX} + d_{ABX} f_{CDX}) c_A d_B d_C d_D \\ \widehat{\text{Tr}}(D_A D_B D_C D_D) &= \frac{1}{2} \delta_{(AB} \delta_{CD)} c_A c_B c_C c_D \\ &\quad + \frac{N}{8} (f_{ADX} f_{BCX} - f_{ABX} f_{CDX} + d_{ABX} d_{CDX} + d_{ADX} d_{BCX}) \eta_{ABCD}\end{aligned}\tag{2.12}$$

where $\eta_{ABCD} = d_A d_B d_C d_D - 8\delta_{A+B+C+D,0}$.

3 Two-, three- and four-point functions at one loop

The Feynman rules are collected in Appendix. Evaluating the one-loop contributions is lengthy but straightforward. In this section we consider the planar part of the 2-, 3- and 4-point functions

and, adopting the dimensional regularization ($\epsilon = 4 - D$, as usual), we extract first the planar part and, out of it, the divergent part. The relevant results are written down below. The 2- and 3-point functions are exactly parallel to the corresponding ones in ordinary gauge theories, and some of them are written down below only for the sake of comparison.

Gluons carry Lorentz indices μ, ν, \dots , color indices A, B, \dots , and momenta p, q, \dots . Ghosts carry only the last two type of labels. All the momenta are entering, unless otherwise specified, and we use the notation $p \times q = \frac{1}{2}p_\mu \theta^{\mu\nu} q_\nu$.

2-point function. We have two nonvanishing contribution to the UV divergent part:

– gluons circulating inside the loop:

$$i \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \delta_{AB} N \left[\frac{19}{12} g_{\mu\rho} p^2 - \frac{11}{6} p_\mu p_\nu \right] \quad (3.13)$$

– ghosts circulating inside the loop:

$$i \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \delta_{AB} N \left[\frac{1}{12} g_{\mu\rho} p^2 + \frac{1}{6} p_\mu p_\nu \right] \quad (3.14)$$

Their sum is:

$$i \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \delta_{AB} N \frac{5}{3} [g_{\mu\rho} p^2 - p_\mu p_\nu] \quad (3.15)$$

which entails the usual renormalization constant

$$Z_3 = 1 + \frac{5}{3} g^2 N \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \quad . \quad (3.16)$$

3-point function. The external gluons carry labels (A, p, μ) , (B, q, ν) and (C, k, λ) for the Lie algebra, momentum and Lorentz indices. They are ordered in anticlockwise sense. The triangle diagram gives

$$\begin{aligned} & - \frac{13}{8} g^3 N \frac{1}{(4\pi)^2} \frac{2}{\epsilon} (\cos(p \times q) f_{ABC} + \sin(p \times q) d_{ABC}) \\ & \cdot ((p - q)_\lambda g_{\mu\nu} + (q - k)_\mu g_{\nu\lambda} + (k - p)_\nu g_{\mu\lambda}) \end{aligned} \quad (3.17)$$

The diagram with one three gluon vertex and one four-gluon vertex gives:

$$\begin{aligned} & \frac{9}{4} g^3 N \frac{1}{(4\pi)^2} \frac{2}{\epsilon} (\cos(p \times q) f_{ABC} + \sin(p \times q) d_{ABC}) \\ & \cdot ((p - q)_\lambda g_{\mu\nu} + (q - k)_\mu g_{\nu\lambda} + (k - p)_\nu g_{\mu\lambda}) \end{aligned} \quad (3.18)$$

The contribution of the two ghost circulating diagrams is:

$$\begin{aligned} & \frac{1}{24} g^3 N \frac{1}{(4\pi)^2} \frac{2}{\epsilon} (\cos(p \times q) f_{ABC} + \sin(p \times q) d_{ABC}) \\ & \cdot ((p - q)_\lambda g_{\mu\nu} + (q - k)_\mu g_{\nu\lambda} + (k - p)_\nu g_{\mu\lambda}) \end{aligned} \quad (3.19)$$

The sum of the coefficients is

$$- \frac{13}{8} + \frac{9}{4} + \frac{1}{24} = \frac{2}{3} \quad (3.20)$$

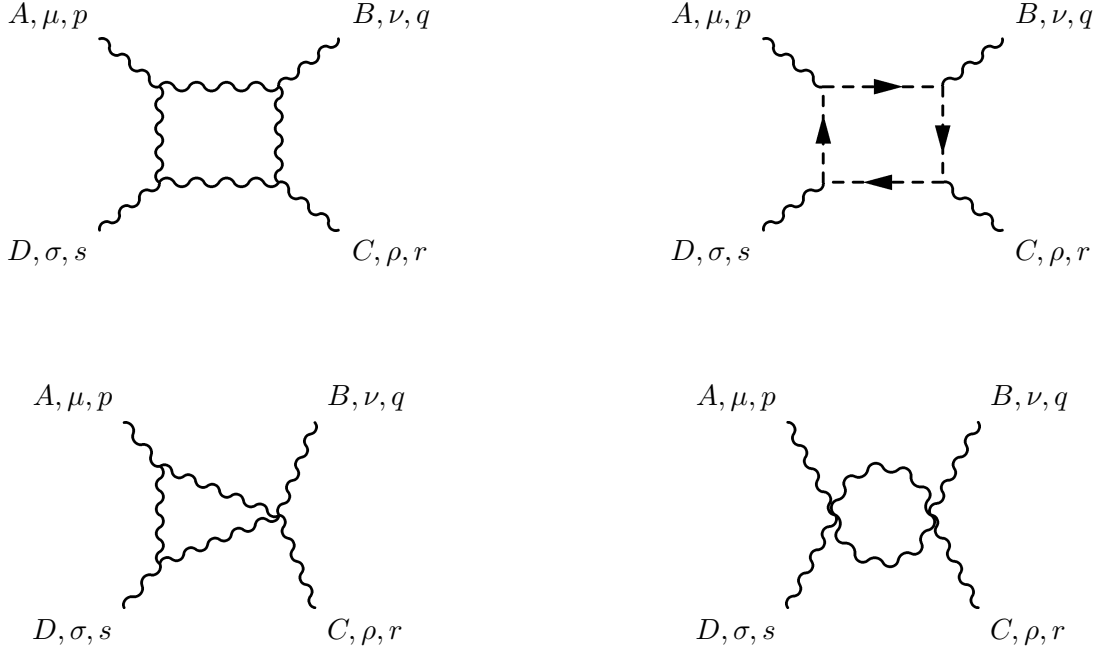


Figure 1: One loop contributions to the four-point function

Therefore, as in the ordinary YM theory, the renormalization constant Z_1 is

$$Z_1 = 1 + \frac{2}{3}g^2 N \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \quad . \quad (3.21)$$

4-point function. The external gluons carry labels (A, μ, p) , (B, ν, q) , (C, ρ, r) and (D, σ, s) for Lie algebra, Lorentz index and momentum, as shown in Figure 1.

There are four distinct graphs contributing to the 4-gluon vertex: the gluon box \mathfrak{b} , the ghost box \mathfrak{g} , the gluon triangle \mathfrak{t} and the gluon candy \mathfrak{c} . There are two main type of contributions, distinguished by their Lie algebra tensor structure. The first is characterized by Kronecker delta functions in the Lie algebra indices, while the second consists of d and f tensors. The first type contributions, which are potentially dangerous for renormalizability, fortunately vanish.

The second type contributions have the general form

$$\begin{aligned} -ig^4 \frac{2}{\epsilon} \frac{1}{(4\pi)^2} \left[\left(\frac{N}{8} \cos(p \times s - q \times r) L_{ABCD} + \frac{N}{8} \sin(p \times s - q \times r) M_{ABCD} \right) K_{\mu\nu\rho\sigma}^i \right. \\ + \left(\frac{N}{8} \cos(p \times r - q \times s) L_{BACD} - \frac{N}{8} \sin(p \times r - q \times s) M_{BACD} \right) K_{\nu\mu\rho\sigma}^i \\ \left. + \left(\frac{N}{8} \cos(p \times s + q \times r) L_{ACBD} + \frac{N}{8} \sin(p \times s + q \times r) M_{ACBD} \right) K_{\mu\rho\nu\sigma}^i \right] \quad (3.22) \end{aligned}$$

where

$$\begin{aligned} L_{ABCD} &= d_{ABX} d_{CDX} + d_{ADX} d_{CBX} - f_{ABX} f_{CDX} + f_{ADX} f_{BCX} \\ M_{ABCD} &= d_{ABX} f_{CDX} + d_{ADX} f_{BCX} + f_{ABX} d_{CDX} - f_{ADX} d_{BCX} \end{aligned} \quad (3.23)$$

and

$$\begin{aligned} K_{\mu\nu\rho\sigma}^{\mathbf{b}} &= \frac{94}{3} g_{\mu\nu} g_{\rho\sigma} + \frac{94}{3} g_{\mu\sigma} g_{\nu\rho} + \frac{34}{3} g_{\mu\rho} g_{\nu\sigma} \\ K_{\mu\nu\rho\sigma}^{\mathbf{g}} &= -\frac{1}{3} (g_{\mu\nu} g_{\rho\sigma} + g_{\mu\sigma} g_{\nu\rho} + g_{\mu\rho} g_{\nu\sigma}) \\ K_{\mu\nu\rho\sigma}^{\mathbf{t}} &= -(46 g_{\mu\nu} g_{\rho\sigma} + 46 g_{\mu\sigma} g_{\nu\rho} - 32 g_{\mu\rho} g_{\nu\sigma}) \\ K_{\mu\nu\rho\sigma}^{\mathbf{c}} &= 16 (7 g_{\mu\nu} g_{\rho\sigma} + 7 g_{\mu\sigma} g_{\nu\rho} - 8 g_{\mu\rho} g_{\nu\sigma}) \end{aligned} \quad (3.24)$$

The 4-point vertex is now easily calculated by summing all the contributions with the appropriate symmetry factors. The contributions (3.22) give rise to

$$\begin{aligned} i \frac{N}{12} g^4 \frac{2}{\epsilon} \frac{1}{(4\pi)^2} & \left[\left(\cos(p \times s - q \times r) L_{ABCD} + \sin(p \times s - q \times r) M_{ABCD} \right) T_{\mu\nu\rho\sigma} \right. \\ & + \left(\cos(p \times r - q \times s) L_{BACD} - \sin(p \times r - q \times s) M_{BACD} \right) T_{\nu\mu\rho\sigma} \\ & \left. + \left(\cos(p \times s + q \times r) L_{ACBD} + \sin(p \times s + q \times r) M_{ACBD} \right) T_{\mu\rho\nu\sigma} \right] \end{aligned} \quad (3.25)$$

where

$$T_{\mu\nu\rho\sigma} = g_{\mu\nu} g_{\rho\sigma} + g_{\mu\sigma} g_{\nu\rho} - 2 g_{\mu\rho} g_{\nu\sigma} \quad (3.26)$$

Comparing (3.25) with eq.(3.32) in the Appendix, we see that the contribution (3.25) implies that the four-A term in the action is renormalized with a Z_4 given by

$$Z_4 = 1 - \frac{1}{3} g^2 N \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \quad . \quad (3.27)$$

This is the same renormalization that occurs in ordinary $U(N)$ Yang-Mills theories. Therefore, *the noncommutative $U(N)$ Yang-Mills theories are one-loop renormalizable.*

The $U(1)$ case must be treated separately. Using the corresponding Feynman rules (see Appendix), one finds the 2- and 3-point contributions evaluated above with $f = 0$ and $d = 1$ and multiplied by $\frac{1}{2}$. As for the 4-point function, the term corresponding to (3.25) is obtained by setting $L = 2$ and $M = 0$ in the latter. Therefore all the renormalization constants satisfy the renormalization conditions, and, as a consequence, the noncommutative $U(1)$ gauge theory is one-loop renormalizable too, [11, 13].

We would like finally to present some results (which are obtained without much effort as byproducts of the previous calculations) concerning a restriction from the $U(N)$ to the $SU(N)$ case. It is not known what a noncommutative $SU(N)$ gauge theory is, although an attempt of defining it has been done recently, [33]. In particular we do not know the explicit form of the action. Therefore we can only try to guess the relevant Feynman rules. The most obvious possibility one can envisage is that they are simply obtained from the Feynman rules of the noncommutative $U(N)$ theory by

restricting everywhere the $U(N)$ indices A, B, \dots to the corresponding $SU(N)$ ones a, b, \dots . As one can see in this case the renormalization constants do not coincide with the ones in the ordinary $SU(N)$ gauge theory. Strictly speaking this is not enough to conclude that the noncommutative $SU(N)$ theory is nonrenormalizable, unless one assumes that the $\theta \rightarrow 0$ limit of the quantum theory is smooth.

However, even allowing for such more general possibility, it is easy to show that the theory defined by such Feynman rules is not one-loop renormalizable, see also [26]. To this purpose it is sufficient to compare the ratio of the renormalization constants of gluon propagator, Z_3 , and the three gluon vertex, Z_1 , with the ratio of ghost propagator, \tilde{Z}_3 , and ghost-ghost-gluon vertex, \tilde{Z}_1 . If the $SU(N)$ theory were renormalizable, we should find $Z_3/Z_1 = \tilde{Z}_3/\tilde{Z}_1$. Instead we obtain

$$\begin{aligned} Z_1 &= 1 + g^2 \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \frac{1}{4} \left(\frac{N^2 - 2}{N} \right) \\ Z_3 &= 1 + g^2 \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \frac{5}{3} \left(\frac{N^2 - 2}{N} \right) \\ \tilde{Z}_1 &= 1 - g^2 \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \frac{1}{2} \left(\frac{N^2 - 3}{N} \right) \\ \tilde{Z}_3 &= 1 - g^2 \frac{1}{(4\pi)^2} \frac{2}{\epsilon} \frac{1}{2} \left(\frac{N^2 - 2}{N} \right) \quad , \end{aligned}$$

where we used the traces over the $SU(N)$ indices that can be found in [30].

Appendix. Feynman rules for noncommutative $U(N)$ theories.

Gluons carry Lorentz indices μ, ν, \dots , color indices A, B, \dots , and momenta p, q, \dots . Ghosts carry only the last two type of labels. All the momenta are entering unless otherwise specified.

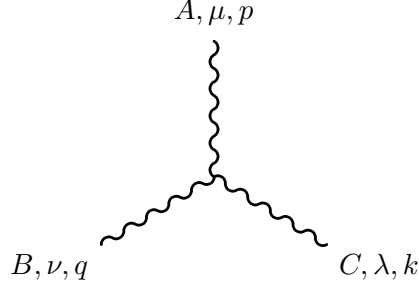
gluon propagator.

$$A, \mu \quad \text{---} \overset{p}{\sim} \quad B, \nu \quad \quad - \frac{i}{p^2} \delta_{AB} g_{\mu\nu} \quad (3.28)$$

ghost propagator.

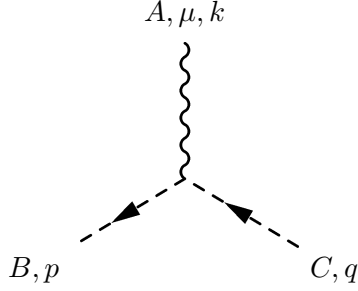
$$A, \mu \quad \text{---} \overset{p}{\rightarrow} \text{---} \quad B, \nu \quad \quad \frac{i}{p^2} \delta_{AB} \quad (3.29)$$

3-gluon vertex. The external gluons carry labels (A, μ, p) , (B, ν, q) and (C, λ, k) for the Lie algebra, momentum and Lorentz indices and are ordered in anticlockwise sense:



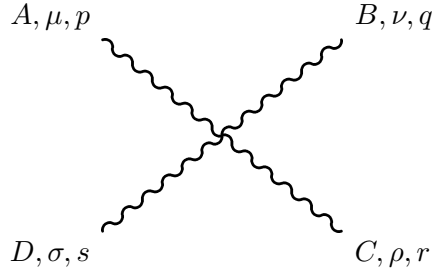
$$-g(f_{ABC} \cos(p \times q) + d_{ABC} \sin(p \times q))(g_{\mu\nu}(p-q)_\lambda + g_{\nu\lambda}(q-k)_\mu + g_{\lambda\mu}(k-p)_\nu) \quad (3.30)$$

ghost vertex. The gluon carries label (A, μ, k) , the ghosts (B, p) and (C, q) :



$$-g p_\mu (f_{ABC} \cos(p \times q) - d_{ABC} \sin(p \times q)) \quad (3.31)$$

4-gluon vertex. The gluons carry labels (A, μ, p) , (B, ν, q) , (C, ρ, r) and (D, σ, s) for Lie algebra, Lorentz index and momentum. They are clockwise ordered:



$$\begin{aligned} -ig^2 \quad & \left[(f_{ABX} \cos(p \times q) + d_{ABX} \sin(p \times q)) \right. \\ & \quad \cdot (f_{XCD} \cos(r \times s) + d_{XCD} \sin(r \times s)) (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\sigma} g_{\nu\rho}) \\ & + (f_{ACX} \cos(p \times r) + d_{ACX} \sin(p \times r)) \\ & \quad \cdot (f_{XDB} \cos(s \times q) + d_{XDB} \sin(s \times q)) (g_{\mu\sigma} g_{\nu\rho} - g_{\mu\nu} g_{\rho\sigma}) \\ & + (f_{ADX} \cos(p \times s) + d_{ADX} \sin(p \times s)) \\ & \quad \cdot (f_{XBC} \cos(q \times r) + d_{XBC} \sin(q \times r)) (g_{\mu\nu} g_{\rho\sigma} - g_{\mu\rho} g_{\nu\sigma}) \left. \right] \end{aligned}$$

With elementary manipulations we can rewrite this as follows:

$$\begin{aligned}
-\frac{i}{4}g^2 \quad & \left[\left(\cos(p \times s - q \times r) L_{ABCD} + \sin(p \times s - q \times r) M_{ABCD} \right) T_{\mu\nu\rho\sigma} \right. \\
& + \left(\cos(p \times r - q \times s) L_{BACD} - \sin(p \times r - q \times s) M_{BACD} \right) T_{\nu\mu\rho\sigma} \\
& \left. + \left(\cos(p \times s + q \times r) L_{ACBD} + \sin(p \times s + q \times r) M_{ACBD} \right) T_{\mu\rho\nu\sigma} \right] \quad (3.32)
\end{aligned}$$

The tensors M, L, T are defined in the text.

The Feynman rules for $U(1)$ are formally obtained from the above ones by setting $\theta = 0$, the tensor $f = 0$ and $d = 1$ (therefore, in particular, $L = 2$, $M = 0$).

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